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by

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# Highway Construction under Referendum and Representative Democracy

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## Abstract

The goal of this paper is to analyze how efficient transportation infrastructure decision-process mechanisms are when their implementation are subjected to voting process. We formulate a simple majority voting model where commuters within a linear city, vote on the level and location of transportation infrastructure. We evaluate three existing decision-process mechanisms: referendum; representative voting; and welfare-maximizing policy not only from its locational but also from the efficient point of view.

*JEL Classification:* H71; H73; R51

*Keywords:* Condorcet voting; referendum; representative democracy; transportation infrastructure

## 1. Introduction

The goal of our study is to contribute to the theoretical literature concerning the welfare effects of voting decision on choosing a highway location. Current efforts of Japanese government to reduce the inefficiencies in transportation policies is the primary motivation in the conduct of this study.

We formulate a simple majority voting model to construct highway in a linear city where residents choose the provision and location of highway through a two-stage voting procedure. In the first stage, voters vote on the highway length. And in the second stage, they vote on its location. This two-stage process seems to be natural because the policies for constructing highway are strongly related to the fiscal policy of governments wherein the length is decided even before the location is chosen.

To determine the level and location of the transportation infrastructure in the world, the following decision-process mechanisms are used: (i) referendum, i.e., all commuters vote directly on the policy; (ii) representative democracy, i.e., the representatives of the districts where commuters are based, choose the policy according to the net benefit of their district; and (iii) welfare-maximizing policy. We compare these mechanisms in the theoretical context.

We now briefly discuss the literature that is related to our study. First, Ohsawa(2000) considered voters that commute from the homes over a linear city to a single point at a central business district (CBD), i.e., multiple origin-single destination trips. Our model differs from this work in at least two respects. The first difference is that our model deals with multiple origin-multiple destination trips, so the individual utility of voters are not necessarily single-peaked. This means that the median voter theorem is of little help. The second difference is that our model analyzes not only referendum but also representative voting. The districts are heterogeneous in that they consist of commuters living in different workplaces. On the other hand, Presson and Tabellini(1994) compared direct and indirect democracy to analyze the process of determining the capital tax rate. Our model, however, explicitly deals with the geographical variables.

The plan of our study is as follows: Section 2.1 states our model. Sections 2.2 and 2.3 characterize voting outcomes under referendum and representative voting mechanisms, respectively. Section 2.4 examines the welfare-maximizing policy. Section 2.5 compares the welfare-maximizing with the voting outcomes. Section 3 discusses the main results. For convenience, all proofs are presented in the Appendix.

## 2. Model

### 2.1. Statement of Model

The model used in this study is similar to that of Ohsawa(2000). There is a line city  $\Omega = [-\frac{l}{2}, \frac{l}{2}]$  where  $O$  is fixed at its center, as shown in Figure 1(a). Our model involves the following assumptions: (a1) the homes and workplaces of voters are uniformly and independently distributed with frequency normalized at unity over the city, irrespective of their commuting distances; (a2) the highway use benefit of the commuters is proportional to the travel distances made; (a3) the highway is provided for free; (a4) a toll-free local road joins any point in the city, and each voter uses highway and/or the toll-free road; (a5) highway construction cost per unit distance is constant; (a6) highway construction is financed through a per-citizen tax.

Assumption (a1) implies that the total volume of trips carried is  $l^2$ . Assumption (a1) also states that the voters are evenly spread with a density  $l$ , so the total voters is  $l^2$ . Assumption (a2) indicates non-rivalry, i.e, no congestion in the highway. Assumption (a5) leads to no economy of scale in highway building. Assumptions (a2) and (a6) means that commuters' preferences are systematically related to their home as well as workplace.

We are concerned with Condorcet winners that defeat all other alternatives under pairwise and simple majority voting rule, as in Cremer et al.(1985), Alesina and Spolaore(1997), Ohsawa(2000). To determine the location of the highway, two-stage voting game is examined. In the first stage, voters select the highway length. After having observed these length, they vote on the highway location. This two-stage process is solved by backward induction. For any choice made in the first stage, the highway location is determined from the corresponding second-stage voting. Accordingly, in the first stage, the voters make their decisions anticipating the consequences of their choices on location competition.

The usual way to describe the trip distribution, i.e., the movement of commuters from one zone to another is to tabulate the number of commuters from one zone to another. We apply this tabulation to the set of commuters that describes a continuum, like Vaughan(1987). As we consider multiple origin-multiple destination trips, the trip distribution has to be defined over two-dimensional space  $[-\frac{l}{2}, \frac{l}{2}] \times [-\frac{l}{2}, \frac{l}{2}]$ , as in Ohsawa(1989). Then any trip between two arbitrarily points within the city is indicated by one point on the two-dimensional space. This is illustrated in Figures 2, 3, and 4 where the horizontal axis measures the location of homes of trips, and the vertical axis measures the location of their workplaces. For example,

the trip starting from home at  $x_1$  (resp.  $x_2$ ) to workplace at  $y_1$  (resp.  $y_2$ ) in Figure 1(b) is represented as a bullet  $(x_1, y_1)$  (resp. a circle  $(x_2, y_2)$ ) in Figure 2. We call this distribution of the homes and workplaces as *trip distribution*, along the lines of Vaughan(1987).

## 2.2. Referendum

We study the referendum, where each commuter votes for the highway that creates the highest *net benefit*, which is defined as his/her benefit for highway service minus a per-citizen tax. Accordingly, voters consider only their own economic well-being, where the source of individual utility is from direct tax and highway travel benefit.

Let  $\beta(> 0)$  be the highway use benefit per unit distance of each commuter instead of the local road. Let  $\gamma(> 0)$  be the highway construction cost per unit distance. Hence, to construct the highway with length  $h$ , a per-citizen tax is imposed on every commuter by  $\frac{\gamma h}{l^2}$ . If  $\frac{\gamma}{l^2} > \beta$ , then the marginal construction cost of each commuter exceeds his marginal benefit. Hence, for all the voters it is worthless to construct highway, so the do-nothing scheme is elected unanimously. Accordingly, we confine our attention to the case of  $\gamma \leq \beta l^2$ .

For the highway location  $R \subseteq \Omega$ , let  $U(R : x, y)$  denote the net benefit of the commuter who travels from the home located at  $x \in \Omega$  to the workplace located at  $y \in \Omega$ .

**Lemma 1** *Any voting solution under this referendum is necessarily symmetrical with respect to the city's center.*

The proof is given in Appendix A.1. This symmetrical solution is justified from the viewpoint of equity. This lemma means that the problem to determine both length and location of highway reduces to the one to fix the length of a continuous and symmetrical highway. Accordingly, policy is one-dimensional in spite of the two-dimensional individual differences.

Let  $R_h$  be the continuous and symmetrical highway with the length  $h$ , i.e.,  $R_h \equiv [-\frac{h}{2}, \frac{h}{2}]$ . Two cases are considered separately corresponding as to whether the commuter passes through the center or not. By using symmetry, without loss of generality, the individual utility  $U(R_h : x, y)$  for any home at  $x$  and any workplace at  $y$  can be expressed either by equation (1) or by equation (2). For  $x \leq 0 \leq y \leq \frac{1}{2}$  with  $|x| \leq y$ , (see the trip from  $x_1$  to  $y_1$  in Figures 1(b) and 2),

$$U(R_h : x, y) = \begin{cases} \beta h - \frac{\gamma}{l^2} h & \text{if } 0 \leq h < 2|x|; \\ \beta \left( \frac{h}{2} - x \right) - \frac{\gamma}{l^2} h & \text{if } 2|x| \leq h < 2y; \\ \beta(y - x) - \frac{\gamma}{3l^2} h & \text{if } 2y \leq h. \end{cases} \quad (1)$$

For  $0 \leq x \leq y \leq \frac{l}{2}$ , (see the trip from  $x_2$  to  $y_2$  in Figures 1(b) and 2),

$$U(R_h : x, y) = \begin{cases} -\frac{\gamma}{l^2}h & \text{if } 0 \leq h < 2x; \\ \beta\left(\frac{h}{2} - x\right) - \frac{\gamma}{l^2}h & \text{if } 2x \leq h < 2y; \\ \beta(y - x) - \frac{\gamma}{l^2}h & \text{if } 2y \leq h. \end{cases} \quad (2)$$

Table 1 gives the tabular expression of the utility  $U(R_h : x, y)$ , and Table 2 express its partial derivative with respect to the highway length  $h$ . Thus, the utility function of a trip is closely associated with the geographical position of its home and workplace. The utility functions  $U(R_h : x_1, y_1)$  and  $U(R_h : x_2, y_2)$  versus the highway length are illustrated as dotted and solid lines, respectively, in Figure 5, where  $\beta = 4$  and  $\gamma = 1$ . As illustrated in  $U(R_h : x_1, y_1)$  in the figure, the utility function (1) for commuters who pass through the center  $O$  is single-peaked with respect to the length  $h$ . However, the utility function (2) for commuters who do not pass through the center cannot always be single-peaked. More precisely, it is bimodal for  $x \neq y$ , as shown in  $U(R_h : x_2, y_2)$  in the figure. Therefore, the median voter theorem presented by Black(1948) is no longer available.

**Proposition 1** *The length of equilibrium under the referendum  $h^*$  is given by*

$$h^* = \begin{cases} \frac{2}{3}l, & \text{if } \frac{1}{5}\beta l^2 \leq \gamma \leq \frac{(3-\sqrt{5})}{2}\beta l^2 (\approx 0.382\beta l^2); \\ 0, & \text{if } \frac{1}{2}\beta l^2 \leq \gamma \leq \beta l^2. \end{cases}$$

The proof is provided in Appendix A.2. The logic behind this proposition is as follows. For low construction costs with  $0 \leq \gamma < \frac{1}{2}\beta l^2$ , the short-distance commuters do not prefer highway extension, since this extension cannot increase their accessibility. So they, who corresponds to diagonally upward elements in Table 3, would unite to discourage highway extension, and the other commuters would unite to support such extension. The equilibrium condition that each united commuters consists of just half the total votes indicates that the length has to be either two thirds of the city length or no change at all. This is illustrated in Figures 3 and 4, respectively, where the area of shaded and unshaded regions are the same. On the other hand, for high costs with  $\frac{1}{2}\beta l^2 < \gamma \leq \beta l^2$ , the commuters who do not pass through the center do not prefer any highway construction. They are indicated by diagonally upward elements in Table 3. The equilibrium condition that each united commuters consists of just half the total votes results in no change at all in the highway length, as shown in Figure 4. As we will see in Appendix A.2., if an equilibrium exists, then the length equilibrium has to be two thirds of the city length for  $0 \leq \gamma \leq \frac{(3-\sqrt{5})}{2}\beta l^2$  and no change at all for  $\frac{(3-\sqrt{5})}{2}\beta l^2 \leq \gamma \leq \beta l^2$ .

However, these solutions may break down. For very low costs with  $0 \leq \gamma < \frac{1}{5}\beta l^2$ , the short-distance commuters near the city's boundary can prefer longer highways than the two

thirds of the city length. So constructing two thirds of the city length for the highway can not be a Condorcet winner. On the other hand, for moderate low costs with  $\frac{(3-\sqrt{5})}{2}\beta l^2 \leq \gamma < \frac{1}{2}\beta l^2$ , long-distance commuters might prefer constructing short highways than the do-nothing scheme. Therefore, the do-nothing scheme can not be Condorcet solution.

### 2.3. Representative Democracy

When there are a large number of voters, democracy has to be a representative voting instead of a referendum. This is because the number becomes too large to assemble directly. Thus, in reality, referendum is very rare. It is therefore natural to analyze representative democracy, where policy choices are delegated to representatives. We examine the representative democracy, where commuters are allocated to  $m$  geographically connected districts of equal size,  $\frac{l}{m}$ , based on their home positions. These districts are numbered in ascending order from the left extremity of the city, as shown in Figure 1(c). For brevity, let  $\omega_{i,m}$ ,  $s_{i,m}$  and  $t_{i,m}$  denote the location of  $i$ -th district, its left and right boundary for the total number of districts  $m$ , i.e.,  $\omega_{i,m} \equiv [s_{i,m}, t_{i,m}] \equiv [(\frac{i-1}{m} - \frac{1}{2})l, (\frac{i}{m} - \frac{1}{2})l]$ .

We assume that representatives choose the highway location that creates the highest *welfare*, that is defined as the total of its members' utility. Hence, the representatives act on the basis of their maximal welfare. This assumption can be justified because each representatives can control the utility of corresponding commuters by selecting a highway that creates the highest net benefit, and by redistributing the earnings among commuters. As the district becomes bigger, the winner approaches the welfare-maximizing solution because removing the heterogeneity between districts would induce voting outcome to be more efficient. However, at the same time, as the district becomes bigger, so does the cost of redistributing the earnings among commuters within the district. Thus, there is trade-off between decreasing the social welfare and increasing the redistribution cost. We characterize this trade-off between election outcomes and district magnitude.

As in the referendum, if  $\gamma > \beta l^2$ , then the do-nothing scheme is selected unanimously. We, therefore, focus our attention to the case of  $\gamma \leq \beta l^2$ . For highway  $R \subseteq \Omega$ , let  $V(R : \omega_{i,m})$  denote the net benefit of the  $i$ -th district. Since the number leaving a given district can be calculated by summing up all trips emanating from that district, we obtain

$$V(R : \omega_{i,m}) = \int_{x \in \omega_{i,m}} \int_{y \in \Omega} U(R : x, y) dy dx. \quad (3)$$

For example,  $V(R : \omega_{7,10})$  and  $V(R : \omega_{9,10})$  can be obtained by integrating  $U(R : x, y)$  over



the diagonally upward and downward sloping striped rectangles, respectively, in Figure 2.

**Lemma 2** *Any voting outcome under this representative democracy is necessarily symmetrical with respect to the city's center.*

The details of this lemma are in Appendix A.3. This lemma states that the problem to determine highway location leads to the one to fix the length of a continuous highway.

Straightforward but complicated manipulation yields that for  $-\frac{l}{2} \leq s < t \leq \frac{l}{2}$ ,

$$V(R_h : [s, t]) = \begin{cases} \frac{(\beta l^2 - 2\gamma)(t-s)}{2l} h, & \text{for } 0 < h \leq 2s; \\ \frac{(\beta l^2 - 2\gamma)(t-s)}{2l} h - \frac{\beta}{12}(h^3 - 3sh^2 + 4s^3), & \text{for } 2s < h \leq 2t; \\ \frac{(\beta l^2 - 2\gamma)(t-s)}{2l} h - \frac{\beta}{12}(t-s)(3h^2 - 4(t-s)^2 - 12st), & \text{for } 2t \leq h \leq l. \end{cases} \quad (4)$$

The welfare  $V(R_h : \omega_{7,10})$  and  $V(R_h : \omega_{9,10})$  are presented as dotted and solid lines, respectively, in Figure 6. As a special case, when the district consists of a point located at  $x$ , then

$$V(R_h : \{x\}) = \begin{cases} \frac{(\beta l^2 - 2\gamma)}{2l} h, & \text{for } 0 < h \leq 2x; \\ \frac{(\beta l^2 - 2\gamma)}{2l} h - \frac{\beta}{4}(h^2 - 4x^2), & \text{for } 2x \leq h \leq l. \end{cases}$$

This implies that  $V(R_h : \{x\})$  increases with the distance from the center, and  $V(R_h : \{x\})$  is downward convex with respect to  $h$ . Combining this observation with equation (3) means that i) outlying districts can obtain higher welfare  $V(R_h : [s, t])$  at the expense of districts near the center; ii) differentiating from the individual utility  $U(R_h : x, y)$ , the welfare  $V(R_h : [s, t])$  is also single-peaked with respect to the length  $h$ .

We concentrate on only the cases of  $m = 2, 6, 10, 14, \dots$ , i.e.,  $m = 4i - 2$  for any natural number  $i$  because the median ideal point is unambiguously defined. Making use of the median voter theorem, the median-voter equilibrium is given by the ideal point of the  $(\frac{3m+2}{4})$ -th district. This ideal point can be derived by differentiating the welfare  $V(R_h : \omega_{\frac{3m+2}{4}, m})$  with respect to  $h$ , and then putting the derivative equal to zero, and solving the resulting equation for  $h$ .

**Proposition 2** *The length of equilibrium under representative democracy for  $m$  districts  $h^\circ(m)$  is given by*

$$h^\circ(m) = \begin{cases} l - \frac{2\gamma}{\beta l}, & \text{if } 0 < \gamma < \frac{(m-2)}{4m} \beta l^2; \\ \frac{(m-2)l + \sqrt{(m-2)^2 l^2 + 32m(l^2 - 2\gamma/\beta)}}{4m}, & \text{if } \frac{(m-2)}{4m} \beta l^2 \leq \gamma < \frac{1}{2} \beta l^2; \\ 0, & \text{if } \frac{1}{2} \beta l^2 \leq \gamma \leq \beta l^2. \end{cases}$$

The proof is given in Appendix A.4. Five length  $h^\circ(2)$ ,  $h^\circ(6)$ ,  $h^\circ(10)$ ,  $h^\circ(14)$ , and  $h^\circ(\infty)$  are presented in Figure 7, respectively. This proposition and this figure give a casual relationship

running from construction cost to length equilibrium. We recognize from the proposition and this figure that the length declines continuously with the cost for  $0 < \gamma < \frac{1}{2}\beta l^2$ , irrespective of the number of districts  $m$ , as we would expect. To be more precise, the length decreases linearly for low construction costs, while it decreases more slowly for high construction costs than for the low ones. When  $\gamma = \frac{1}{2}\beta l^2$ , then  $h^\circ(m) = \frac{m-2}{2m}l = \frac{1}{2}s_{\frac{3m+2}{4},m}$ . Accordingly, in the case of  $m > 2$ , however, the length equilibrium jumps to zero at  $\gamma = \frac{1}{2}\beta l^2$ .

There are four points worth noting. First, this proposition implies that  $\frac{h^\circ(m)}{2} \geq s_{\frac{3m+2}{4},m}$ . This would mean that if highway is constructed through voting, then it has to exceed the boundary near to the center  $O$  of the median districts  $\omega_{\frac{m+2}{4},m}$  and  $\omega_{\frac{3m+2}{4},m}$ , that is to say, it covers completely the districts  $\omega_{\frac{m+2}{4}+1,m}, \dots, \omega_{\frac{3m+2}{4}-1,m}$ .

Second, it follows from Proposition 2 that in the smallest district case, i.e.,  $m = \infty$ ,  $h^\circ(\infty) \leq \frac{1}{2}l$  for  $0 < \gamma < \frac{\beta l^2}{2}$ . On the other hand, consider a simple optimization problem to locate two point facilities in the interval  $\Omega$  such that the total travel cost from commuters' homes to either facilities is smallest. The optimal solutions are given by  $-\frac{l}{4}$  and  $\frac{l}{4}$ . Interestingly, this proposition means that any length equilibrium is inside quartiles indicating that it has to be inside the segment between these solutions.

Third, we show an example that the voting outcome may not be the solution where both the length and location of the highway are simultaneously voted on. If  $l = 1$ ,  $m = \infty$  and  $\beta = 1$ , then the welfare for the highway  $[a, b]$  at the point  $-\frac{1}{2} \leq x \leq \frac{1}{2}$  is:

$$V([a, b] : \{x\}) = \begin{cases} \frac{1}{2}((b-a)^2 + (1-2b)(b-a)) - \gamma(b-a), & \text{for } -\frac{1}{2} < x \leq a; \\ \frac{1}{2}((x-a)(a+x+1) + (b-x)(1-b-x)) - \gamma(b-a), & \text{for } a \leq x \leq b; \\ \frac{1}{2}((b-a)^2 + (1+2a)(b-a)) - \gamma(b-a), & \text{for } b < x \leq \frac{1}{2}. \end{cases}$$

There is a U-shaped relationship between district position and welfare. If  $\gamma = \frac{15}{31}$ , then  $h^\circ(\infty) = R_{\frac{1}{2}}$ . However, a cycle appears over the three alternatives  $[-\frac{1}{8}, 0]$ ,  $R_{\frac{1}{2}}$  and  $R_{\frac{1}{8}}$ . As shown in Figure 8,  $[-\frac{1}{8}, 0]$  beats  $R_{\frac{1}{2}}$ ,  $R_{\frac{1}{2}}$  beats  $R_{\frac{1}{8}}$ , and  $R_{\frac{1}{8}}$  beats  $[-\frac{1}{2}, 0]$ , meaning that no Condorcet winner exists.

Finally, it is interesting to observe that the voting outcome may please more than half of the districts. It follows from the welfare (4) that the ideal point of the district at  $x$  can be  $l - \frac{2\gamma}{\beta l}$ . Thus, combining this with Proposition 2 yields that the Condorcet winner pleases the ideal point of  $l - \frac{2\gamma}{\beta l}$  locations, more than half districts.

#### 2.4. Social Optimum

In a planning issue, highway is built such that the total net benefits of commuters is max-

imized. If the city is divided into two districts, then the corresponding Condorcet winner coincides with the welfare-maximizing policy. The substitution of  $m = 2$  into Proposition 2 yields

**Proposition 3** *The welfare-maximizing length  $h^*$  is given by*

$$h^* = \begin{cases} \sqrt{l^2 - 2(\gamma/\beta)}, & \text{if } 0 < \gamma < \frac{1}{2}\beta l^2; \\ 0, & \text{if } \frac{1}{2}\beta l^2 \leq \gamma \leq \beta l^2. \end{cases}$$

In comparison with the result in multiple origin-single destination trips in Ohsawa(2000), the optimal length here contains square roots, so it is concave downward with respect to the construction costs. This may reflect multiple origin-multiple destination trips.

This proposition can be illustrated in another way. The flow of volume distribution crossed at the point is decreasing with its distance to the center. This means that the social optimum location has to be a single piece and symmetrical with respect to the center. Substituting  $m = 2$  into social welfare (4) yields the following welfare-maximizing problem:

$$\max_{h \geq 0} W(h) \equiv 2V(R_h : [0, \frac{l}{2}]) = \beta \left( \frac{3l^2 h - h^3}{6} \right) - \gamma h. \quad (5)$$

Since the first term on the right-hand side is the total benefit due to highway service and the second term is the total cost, the optimal length is determined based on the trade off between the marginal benefit and cost. In differentiating  $W(h)$  with respect to  $h$  and placing the derivative equal to zero, and then solving the resulting equation for  $h$  yields Proposition 3.

We see from Proposition 2, 3 and Figure 7 that enlarging the districts shifts the Condorcet winner to the welfare-maximizing length. However, the Condorcet winners and welfare-maximizing solution significantly differ even for small number of districts. Thus, we can analytically characterizes the trade-off between the size of redistribution programs and the electoral outcomes.

In addition, on referring to Figure 7, the voting mechanisms supplies shorter highway than the welfare-maximizing policy for low construction costs, and it also provides longer highway for high construction costs. It is straightforward to verify that if  $\gamma = \frac{3}{8}\beta l^2 = 0.375\beta l^2$ , then  $h^\circ(m) = h^*$ , irrespective of  $m$ . Also, routine calculation shows that for  $\frac{(m-2)}{4m}\beta l^2 \leq \gamma < \frac{1}{2}\beta l^2$ ,  $\frac{d}{d\gamma} h^\circ(m) = -\frac{8}{\beta \sqrt{(m-2)^2 l^2 + 32m(l^2 - 2\gamma/\beta)}}$ . This indicates that  $\frac{d}{d\gamma} h^\circ(m)$  is increasing in  $m$ , i.e.,  $h^\circ(m)$  becomes flatter with  $m$ . Combining these two results means that  $h^\circ(m)$  gets away from the welfare maximizing length  $h^*$  with  $m$ .

### 2.5. Comparison

Along the study of Hansen and Thisse (1981), Labbé (1985), length of equilibrium and welfare-maximizing length are compared. The length under the referendum  $h^*$  in Proposition 1, the length under representative democracy  $h^o(\infty)$  in Proposition 2, and the welfare-maximizing length  $h^*$  in Proposition 3 for  $0 < \gamma \leq \frac{1}{2}\beta l^2$  are shown in Figure 9, using thick, thin, and broken segments, respectively. Note that these three lengths are the same for  $\frac{1}{2}\beta l^2 \leq \gamma$ . This figure shows how the length equilibrium is still short of the welfare-maximizing policy. We see from this figure that i) although Condorcet winner always exists under representative democracy, no winner may exist under the referendum; and ii) although the length equilibrium under the representative democracy depends on the construction cost, the length under the referendum is independent of it. Thus, referendum differs entirely from the representative democracy. This is illustrated in the following points. The length equilibrium under referendum is said to be restricted to two options, i.e., no change at all or two thirds of the city length. The nonexistence of Condorcet winners results from this limited options. On the other hand, under the representative democracy, since the welfare of the median district contains the benefit enjoyed by the commuters going to their workplaces over the city, the length equilibrium changes according to the change in construction cost.

Next, let us compare these outcomes based from viewpoint of social welfare. The social welfare  $W(h^*)$ ,  $W(h^o(\infty))$ , and  $W(h^*)$  can be evaluated by substituting the results in Propositions 1, 2 and 3 into the social welfare (5). They are plotted in Figure 10 with respect to the construction cost using thin, thick and broken segments, respectively, as in Figure 9. Naturally,  $W(h^*)$  is above  $W(h^*)$  and  $W(h^o(\infty))$ , irrespective of construction costs. This figure reveals how the voting processes are inefficient in the provision of highway construction. In particular,  $W(h^o(\infty)) < 0$  for large construction costs.

### 3. Conclusions and Future Studies

Transportation infrastructure resulting from voting process is of interest from policy maker's viewpoint. We thus, formulated a majority voting model where commuters within a linear city vote on the level and location of transportation infrastructure. Then, we evaluated Condorcet winners under the process of referendum and of representative democracy not just from the locational but also from the efficiency point of view.

The main result of our study are threefold. First, we proved that when the construction

cost is expensive, the voting generates a decision in which longer highway is constructed than the welfare-maximizing length. Likewise, when the construction cost is cheap, shorter highway than the welfare-maximizing length is constructed. Second, we demonstrated that highway construction under referendum significantly differs with that under representative democracy. Finally, we revealed how much the welfare under representative democracy is improved, as the districts become larger.

For future studies, it would be interesting to examine geographically discontinuous districts such as Gerrymanders.

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## Mathematical Appendix

### A.1. Proof of Lemma 1

First, let us verify that any discontinuous alternative  $R$  is defeated by the symmetrical continuous alternative  $R_{\frac{3l}{8}}$ . Two alternatives  $R = [-\frac{l}{4}, -\frac{l}{16}] \cup [\frac{l}{16}, \frac{l}{4}]$  and  $R_{\frac{3l}{8}}$  are compared in Figure 11. This figure indicates the domain with  $U(R : x, y) < U(R_{\frac{3l}{8}} : x, y)$ , and the one with  $U(R_{\frac{3l}{8}} : x, y) > U(R : x, y)$  as the diagonally upward and downward sloping striped regions, respectively. We recognize from this figure that the area with  $U(R_{\frac{3l}{8}} : x, y) < U(R : x, y)$  is greater than the area with  $U(R : x, y) > U(R_{\frac{3l}{8}} : x, y)$  by the two shaded squares. The same holds for any discontinuous alternative.

Second, let us prove that any asymmetrical continuous alternative  $R$  is defeated by the symmetrical continuous alternative  $R_{\frac{3l}{8}}$ . Two alternatives  $R = [-\frac{l}{4}, \frac{l}{8}]$  and  $R_{\frac{3l}{8}}$  are compared in Figure 12, where the domain with  $U(R : x, y) < U(R_{\frac{3l}{8}} : x, y)$  and the one with  $U(R_{\frac{3l}{8}} : x, y) > U(R : x, y)$  in Figure 12 are indicated by the diagonally upward and downward sloping striped regions, respectively. We see from this figure that the area with  $U(R_{\frac{3l}{8}} : x, y) < U(R : x, y)$  is greater than the area with  $U(R : x, y) > U(R_{\frac{3l}{8}} : x, y)$  by the two shaded rectangles. The same holds for any asymmetrical continuous alternative.  $\square$

### A.2. Proof of Proposition 1

First, let us demonstrate that if a Condorcet winner exists, it has to be either i)  $R_{\frac{2l}{3}}$  for  $0 \leq \gamma \leq \frac{1}{2}\beta l^2$ ; or ii) the do-nothing scheme for  $\frac{1}{2}\beta l^2 \leq \gamma \leq \beta l^2$ . It follows from Table 3 that if  $0 < \gamma \leq \frac{1}{2}\beta l^2$ , then  $|\{(x, y) \in \Omega^2 | \frac{d}{dh}U(R_h : x, y) \geq 0\}| = \frac{l^2}{2} \Leftrightarrow |\{(x, y) \in \Omega^2 | x, y \leq -\frac{h}{2}\}| + |\{(x, y) \in \Omega^2 | -\frac{h}{2} \leq x, y \leq \frac{h}{2}\}| + |\{(x, y) \in \Omega^2 | \frac{h}{2} \leq x, y\}| = \frac{l^2}{2} \Leftrightarrow h = 0$  or  $h = \frac{2}{3}l$ . Simple calculation shows that  $|\{(x, y) \in \Omega^2 | U(R_{\frac{2l}{3}} : x, y) \geq U(\phi : x, y)\}| \geq \frac{l^2}{2} \Leftrightarrow |\{(x, y) \in \Omega^2 | R_{\frac{2l}{3}} : x, y \geq 0\}| \geq \frac{l^2}{2} \Leftrightarrow \gamma \leq \frac{3-\sqrt{5}}{2}\beta l^2$ . Therefore, if  $0 < \gamma \leq \frac{3-\sqrt{5}}{2}\beta l^2$ , then  $h^* = \frac{2}{3}l$ ; and if  $\frac{3-\sqrt{5}}{2}\beta l^2 \leq \gamma \leq \frac{1}{2}\beta l^2$ , then  $h^* = 0$ , provided that a Condorcet winner exists.

It follows from Table 4 that if  $\frac{1}{2}\beta l^2 \leq \gamma \leq \beta l^2$ , then  $|\{(x, y) \in \Omega^2 | \frac{d}{dh}U(R_h : x, y) \geq 0\}| = \frac{l^2}{2} \Leftrightarrow |\{(x, y) \in \Omega^2 | x \leq -\frac{h}{2}, \frac{h}{2} \leq y\}| + |\{(x, y) \in \Omega^2 | y \leq -\frac{h}{2}, \frac{h}{2} \leq x\}| = \frac{l^2}{2} \Leftrightarrow h = 0$ . Hence  $h^* = 0$ , provided that a Condorcet solution exists.

Next, let us examine the existence of Condorcet winners. Consider the length  $R_0$  for  $\frac{3-\sqrt{5}}{2}\beta l^2 \leq \gamma \leq \beta l^2$ . Note that  $\Delta(h, \gamma) \equiv |\{(x, y) | U(R_h : x, y) - U(R_0 : x, y)\}| - \frac{l^2}{2} = \left(\frac{2\gamma}{\beta}\right)h - \left(\frac{\gamma}{\beta}\right)^2 h^2 + 2\left(\frac{l-h}{2}\right)^2 - \frac{l^2}{2} = h \left[ \left(1 - \frac{2\gamma}{\beta^2}\right) \frac{h}{2} - \left(1 - \frac{2\gamma}{\beta}\right)l \right]$ . There are three cases, depending on  $\gamma$ . If  $\frac{3-\sqrt{5}}{2}\beta l^2 \leq \gamma \leq \frac{1}{2}\beta l^2$ , then  $\Delta(h, \gamma) \leq 0 \Leftrightarrow h \leq \frac{\beta l(\beta l - 2\gamma)}{\beta^2 l^2 - 2\gamma^2}l$ . Therefore,  $R_h$

with  $h \leq \frac{\beta l(\beta l - 2\gamma)}{\beta^2 l^2 - 2\gamma^2} l$  beats  $R_0$ , so  $R_0$  can not be Condorcet winner. If  $\frac{1}{2}\beta l^2 \leq \gamma \leq \frac{\sqrt{2}}{2}\beta l^2$ , then  $\Delta(h, \gamma) \leq 0 \Leftrightarrow h = 0$ , so  $R_0$  beats  $R_h$  for any  $h$ . If  $\frac{\sqrt{2}}{2}\beta l^2 \leq \gamma \leq \beta l^2$ , then  $\Delta(h, \gamma) \leq 0 \Leftrightarrow h \geq \frac{\beta l(2\gamma - \beta l)}{2\gamma^2 - \beta^2 l^2} l \geq l$ , so  $R_0$  beats  $R_h$  for any  $h(\leq h)$ .

Finally, let us examine  $R_{\frac{2l}{3}}$  for  $0 < \gamma \leq \frac{3-\sqrt{5}}{2}\beta l^2$ . If  $0 < \gamma \leq \frac{3-\sqrt{5}}{2}\beta l^2$ , then  $\Delta(h, \gamma) \equiv |\{(x, y) | U(R_h : x, y) \geq U(R_{\frac{2l}{3}} : x, y)\}| - \frac{l^2}{2} = \left(\frac{5\gamma}{3\beta}\right) \left(h - \frac{l^2}{3}\right) - \frac{17}{36l^2} + \frac{1}{4}h^2 - l\frac{h}{2} - \frac{l^2}{2} = (3h - 2l) \left(3h - 4l + 20\frac{\gamma}{\beta}\right)$ . Hence, the inequality  $\Delta(h, \gamma) > 0$  for any  $h$  creates a range of  $\gamma$  for which an Condorcet winner exists. Therefore, if  $\frac{1}{10}\beta l \leq \gamma$ , then  $R_{\frac{2l}{3}}$  is equilibrium. Otherwise, no Condorcet winner exists.  $\square$

### A.3. Proof of Lemma 2

Without losing generality, throughout the appendix, we assume that  $\beta = 1$  and  $l = 1$ .

First, for a given candidate  $R \subseteq \Omega$  we verify that in the case of  $m = \infty$ , there exists a candidate  $\hat{R}$  with  $|\hat{R}| = |R|$  and  $V(\hat{R} : \{x\}) \geq V(R : \{x\})$  for  $\forall x \in \Omega$  such that 1)  $\hat{R}$  is a single piece, or 2)  $\hat{R} \subseteq [-\frac{l}{2}, 0]$  ( $\hat{R} \subseteq [0, \frac{l}{2}]$ ).

Consider two disjoint pieces of  $R$ ,  $R_1 \equiv [g_1 - \frac{c_1}{2}, g_1 + \frac{c_1}{2}]$  and  $R_2 \equiv [g_2 - \frac{c_2}{2}, g_2 + \frac{c_2}{2}]$  such that  $g_1 - \frac{c_1}{2} < 0 < g_2 + \frac{c_2}{2}$ . For  $\delta \geq 0$ , define  $g'_1 \equiv g_1 + \delta$ ,  $g'_2 \equiv g_2 - \frac{c_1}{c_2}\delta$ ,  $R'_1 \equiv [g'_1 - \frac{c_1}{2}, g'_1 + \frac{c_1}{2}]$ ,  $R'_2 \equiv [g'_2 - \frac{c_2}{2}, g'_2 + \frac{c_2}{2}]$  and  $\bar{c} \equiv c_1 + c_2$ , respectively. Note that if  $\delta = 0$ , then  $g'_1 = g_1$ ,  $g'_2 = g_2$ ,  $R'_1 = R_1$  and  $R'_2 = R_2$ . In addition, the centroid of  $R'_1$  and  $R'_2$  coincides with that of  $R_1$  and  $R_2$ , i.e.,  $\bar{g} \equiv \frac{c_1 g_1 + c_2 g_2}{\bar{c}}$ . Then, as  $\delta$  increases,  $g'_1$  increases and  $g'_2$  decreases, so both  $R'_1$  and  $R'_2$  move towards  $\bar{g}$ . Accordingly, it is sufficient to show that  $V(R'_1 \cup R'_2 : \{x\}) \geq V(R_1 \cup R_2 : \{x\})$ .

It follows from welfare (4) that

$$V(R'_1 \cup R'_2 : \{x\}) = \begin{cases} (\frac{\bar{c}}{2} - \bar{c}\bar{g}) - \gamma\bar{c} & \text{for } -\frac{1}{2} \leq x \leq g'_1 - \frac{c_1}{2}; \\ (\frac{\bar{c}}{2} - \frac{c_1^2}{4} - g_1'^2 - c_2 g_2' + x^2) - \gamma\bar{c} & \text{for } g'_1 - \frac{c_1}{2} \leq x < g'_1 + \frac{c_1}{2}; \\ (\frac{\bar{c}}{2} + c_1 g'_1 - c_2 g'_2) - \gamma\bar{c} & \text{for } g'_1 + \frac{c_1}{2} \leq x < g'_2 - \frac{c_2}{2}; \\ (\frac{\bar{c}}{2} - \frac{c_2^2}{4} - g_2'^2 + c_1 g'_1 + x^2) - \gamma\bar{c} & \text{for } g'_2 - \frac{c_2}{2} \leq x < g'_2 + \frac{c_2}{2}; \\ (\frac{\bar{c}}{2} + \bar{c}\bar{g}) - \gamma\bar{c} & \text{for } g'_2 + \frac{c_2}{2} \leq x \leq \frac{1}{2}. \end{cases}$$

Differentiating  $V(R'_1 \cup R'_2 : \{x\})$  with respect to  $\delta$  yields

$$\frac{d}{d\delta} V(R'_1 \cup R'_2 : \{x\}) = \begin{cases} -2(g'_1 - \frac{c_1}{2}) & \text{for } g'_1 - \frac{c_1}{2} \leq x < g'_1 + \frac{c_1}{2}; \\ 2c_1 & \text{for } g'_1 + \frac{c_1}{2} \leq x < g'_2 - \frac{c_2}{2}; \\ \frac{2c_1}{c_2}(g'_2 + \frac{c_2}{2}) & \text{for } g'_2 - \frac{c_2}{2} \leq x < g'_2 + \frac{c_2}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

Hence,  $\frac{d}{d\delta} V(R'_1 \cup R'_2 : \{x\}) \geq 0$  because  $c_1 > 0$ ,  $g'_1 - \frac{c_1}{2} \leq g_1 - \frac{c_1}{2} < 0$  and  $g'_2 + \frac{c_2}{2} \geq g_2 + \frac{c_2}{2} > 0$ .

Thus, we see that for  $\forall x \in \Omega$ ,  $V(R'_1 \cup R'_2 : \{x\})$  is not decreasing in  $\delta$ .

Second, for a given candidate  $R$ , we prove that  $R_{|R|}$  defeats  $R$  in the case of  $m = \infty$ . Based on the first claim, it is sufficient to show that  $R_{|R|}$  defeats  $\hat{R}$ . Define  $a \equiv |R|$ . Let  $\hat{g}$  be the centroid of  $\hat{R}$ . Without loss of generality, we may assume that  $\hat{g} < 0$ . Our analysis can be divided into two cases: *case 1*:  $\hat{R} \subseteq [-\frac{1}{2}, 0]$ ; *case 2*:  $\hat{R}$  is a single piece.

In the first case, we get that for  $-\frac{a}{2} \leq \forall x$ ,  $V(\hat{R} : \{x\}) \leq V(\hat{R} : -\frac{a}{2}) = (\frac{a}{2} - \frac{a^2}{4} - \hat{g}^2 + \frac{a^2}{4}) - \gamma a < (\frac{a}{2} - \frac{a^2}{4}) - \gamma a = V(R_a : 0) \leq V(R_a : \{x\})$ , where the second inequality strictly holds since  $\hat{g} < -\frac{a}{2}$ . This means that  $R_a$  defeats  $\hat{R}$ . In the second case, it follows from  $\hat{g} < 0$  that for  $\frac{a}{2} \leq \forall x$ ,  $V(\hat{R} : \{x\}) = (\frac{a}{2} + a\hat{g}) - \gamma a < (\frac{a}{2}) - \gamma a = V(R_a : \{x\})$ . For  $\hat{g} + \frac{a}{2} \leq \forall x \leq \frac{a}{2}$ ,  $V(\hat{R} : \{x\}) = (\frac{a}{2} + a\hat{g}) - \gamma a < (\frac{a}{2} - \frac{a^2}{4} + (\hat{g} + \frac{a}{2})^2) - \gamma a = V(R_a : \hat{g} + \frac{a}{2}) \leq V(R_a : \{x\})$ . For  $-\frac{a}{2} \leq \forall x \leq \hat{g} + \frac{a}{2}$ ,  $V(\hat{R} : \{x\}) = (\frac{a}{2} - \frac{a^2}{4} - \hat{g}^2 + x^2) - \gamma a < (\frac{a}{2} - \frac{a^2}{4} + x^2) - \gamma a = V(R_a : \{x\})$ . Thus,  $V(\hat{R} : \{x\}) < V(R_a : \{x\})$  for  $-\frac{a}{2} \leq \forall x$ , meaning that  $R_a$  defeats  $\hat{R}$ .

Finally, it follows from equation (3) that this lemma holds for any number of districts  $m$ .

□

#### A.4. Proof of Proposition 2

Differentiating  $V(R_h : [s, t])$  one and two times with respect to  $h$  yields

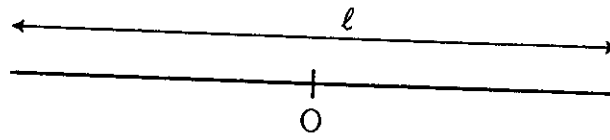
$$\begin{aligned} \frac{\partial V(R_h : [s, t])}{\partial h} &= \begin{cases} \frac{(\beta l^2 - 2\gamma)(t-s)}{2l}, & \text{for } 0 < h \leq 2s; \\ \frac{(\beta l^2 - 2\gamma)(t-s)}{2l} - \frac{\beta}{4}(h^2 - 2sh), & \text{for } 2s < h \leq 2t; \\ \frac{(\beta l^2 - 2\gamma)(t-s)}{2l} - \frac{\beta}{2}(t-s)h, & \text{for } 2t \leq h \leq l; \end{cases} \\ \frac{\partial^2 V(R_h : [s, t])}{\partial h^2} &= \begin{cases} 0, & \text{for } 0 < h \leq 2s; \\ -\frac{\beta}{2}(h - 2s), & \text{for } 2s < h \leq 2t; \\ -\frac{\beta}{2}(t - s), & \text{for } 2t \leq h \leq l. \end{cases} \end{aligned}$$

Solving  $\frac{\partial V(R_h : [s, t])}{\partial h} = 0$  yields

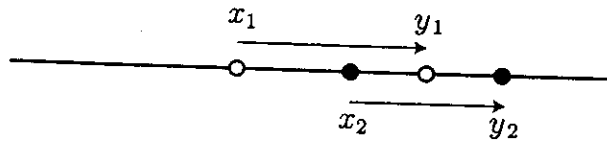
$$h = \begin{cases} l - \frac{2\gamma}{\beta l}, & \text{if } 0 < \gamma < \left(\frac{1}{2} - t\right) \beta l^2; \\ s + \sqrt{s^2 + 2(t-s)(l^2 - 2\gamma/\beta)/l}, & \text{if } \left(\frac{1}{2} - t\right) \beta l^2 \leq \gamma < \frac{1}{2} \beta l^2; \\ 0, & \text{if } \frac{1}{2} \beta l^2 \leq \gamma \leq \beta l^2. \end{cases}$$

It follows from the median voter theorem that the length equilibrium is given by the ideal point of the  $\left(\frac{3m+2}{4}\right)$ -th district. Substituting  $s = \frac{m-2}{4m}l$  and  $t = \frac{m+2}{4m}l$  into the above equation yields the required equation. □

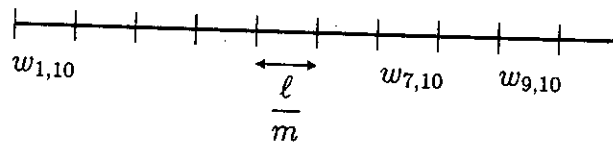




(a)



(b)



(c)

Figure 1: Study Area

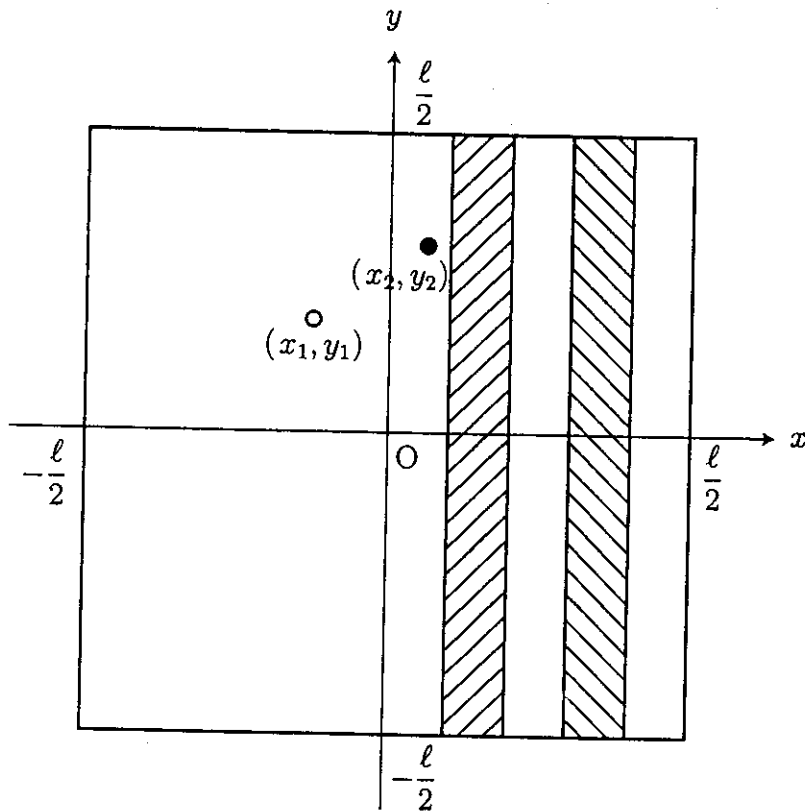


Figure 2: Trip Distribution

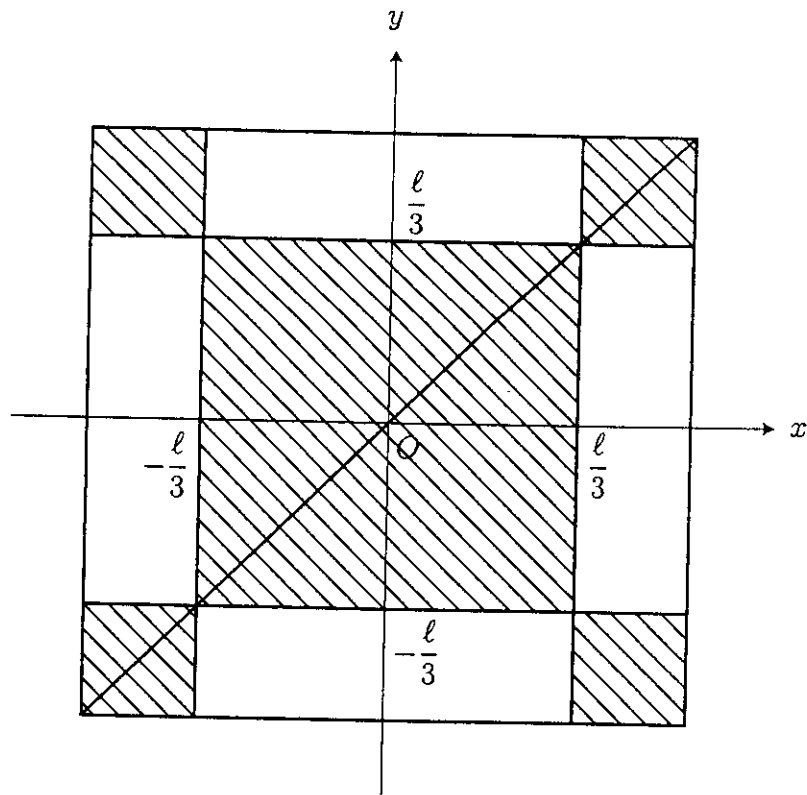


Figure 3: Two Thirds of City Length Scheme

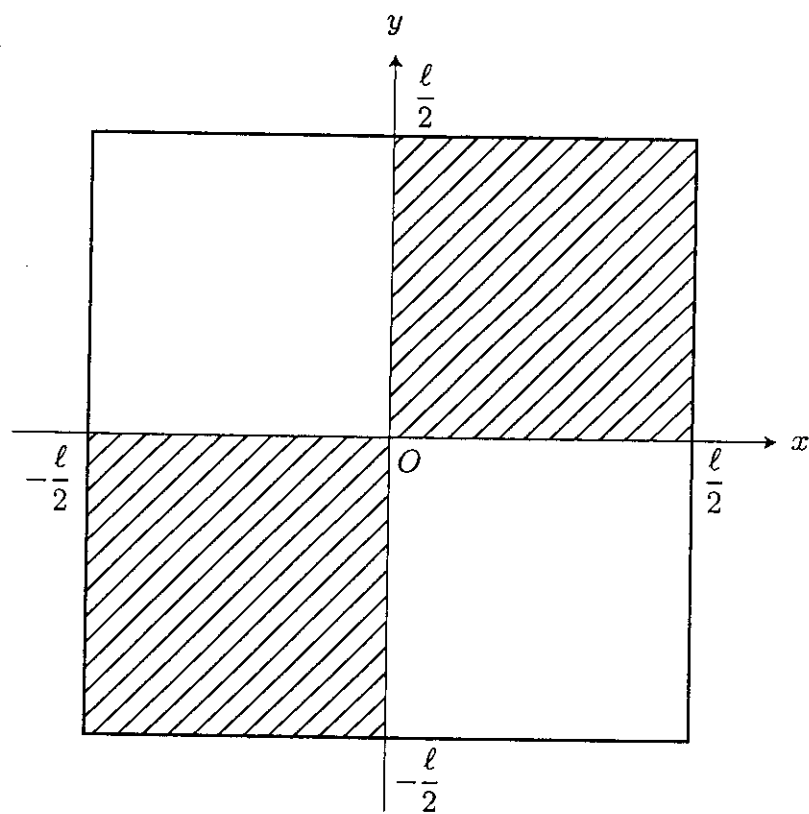


Figure 4: Do-Nothing Scheme

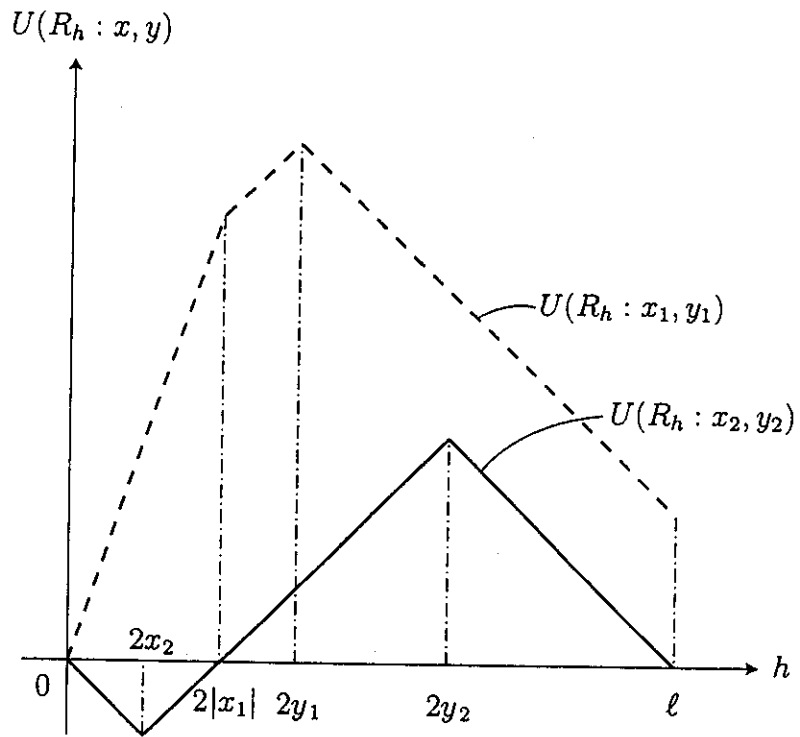


Figure 5: Utility Functions under Referendum

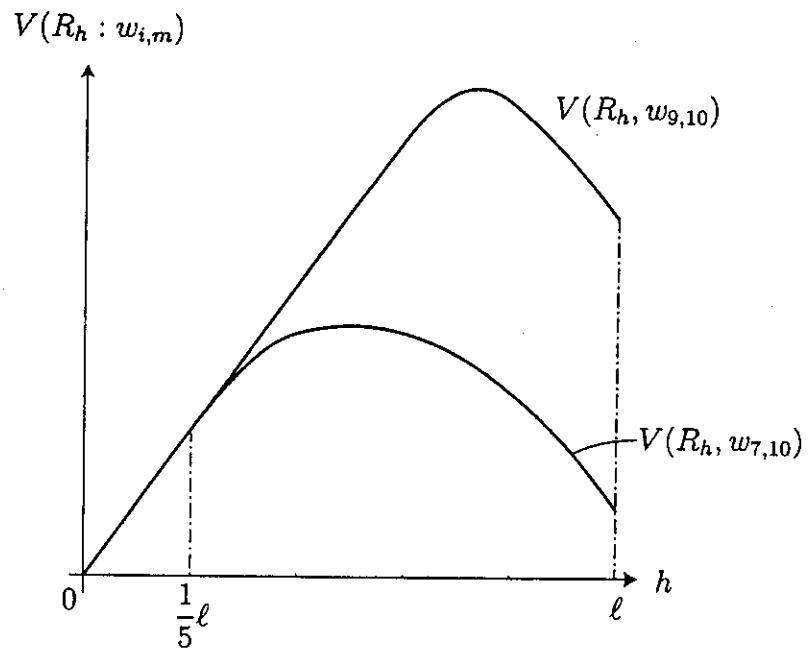


Figure 6: Welfare Functions under Representative Democracy

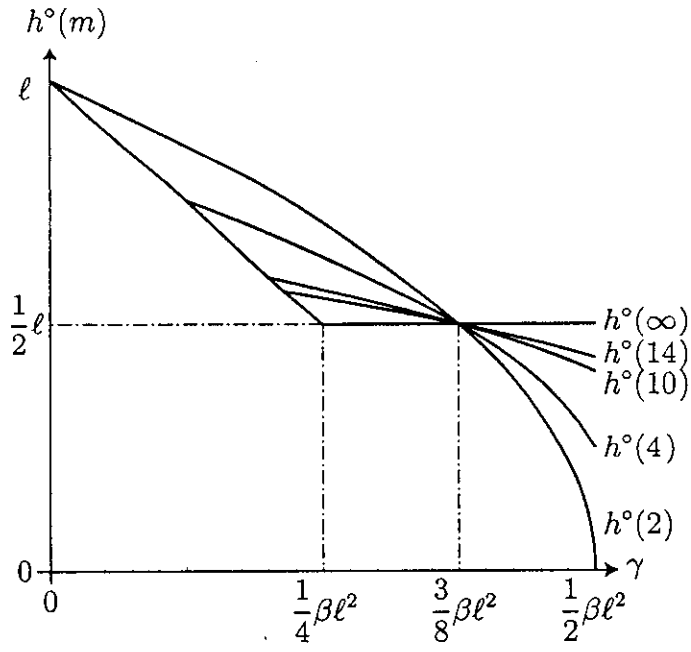


Figure 7: Comparison of Condorcet Winners under Representative Democracy

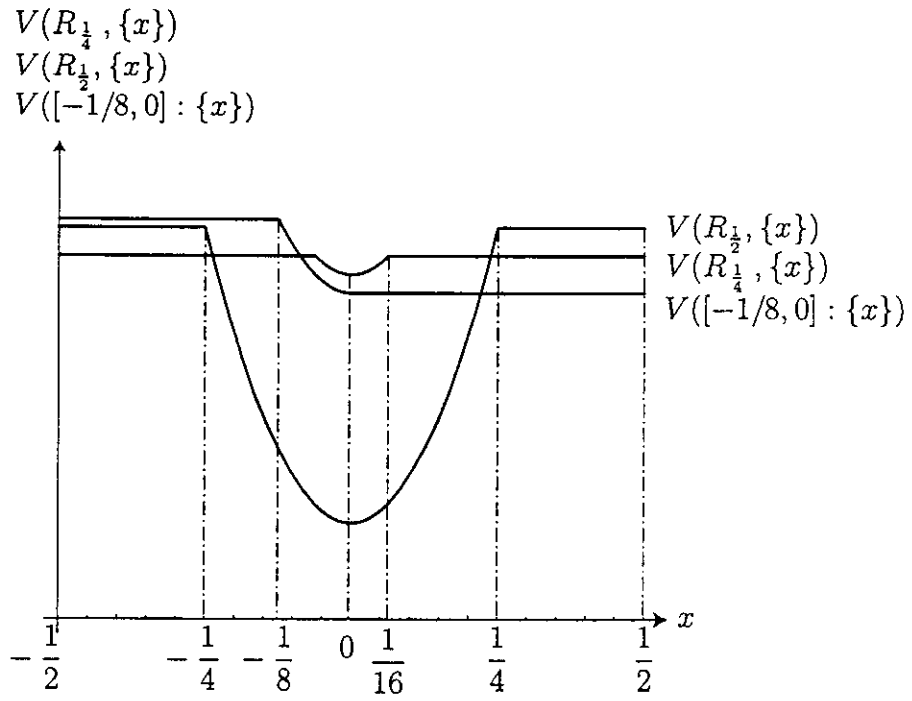


Figure 8: Cycle over Three Alternatives

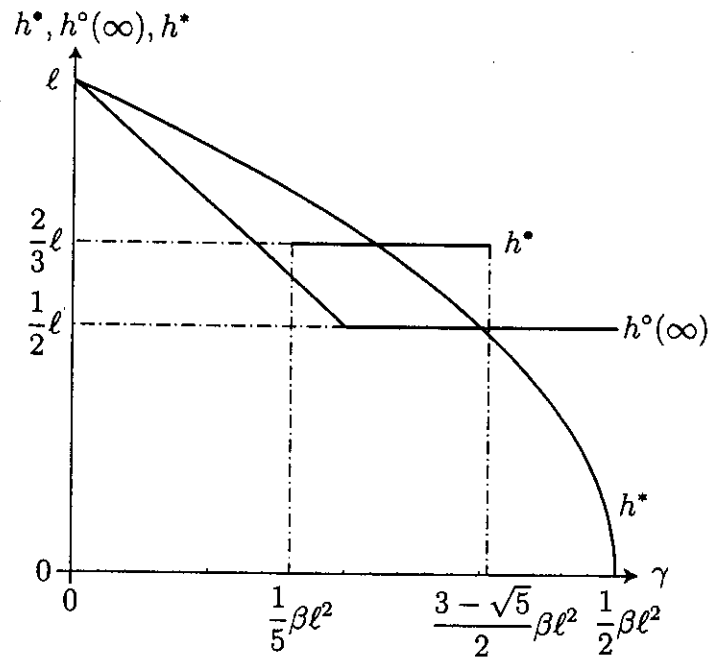


Figure 9: Comparison of Outcomes Based on Location

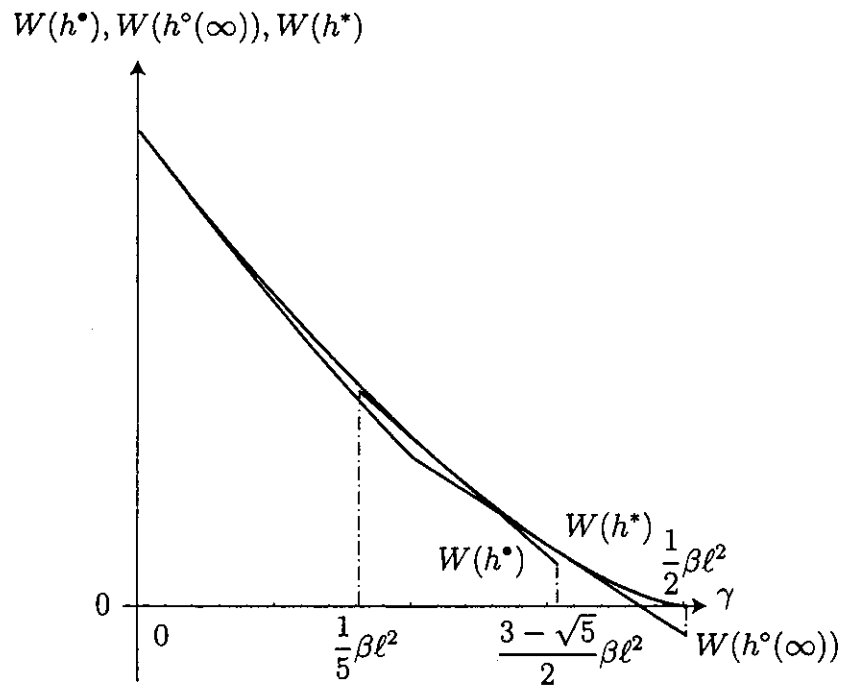


Figure 10: Comparison of Outcomes Based on Social Welfare

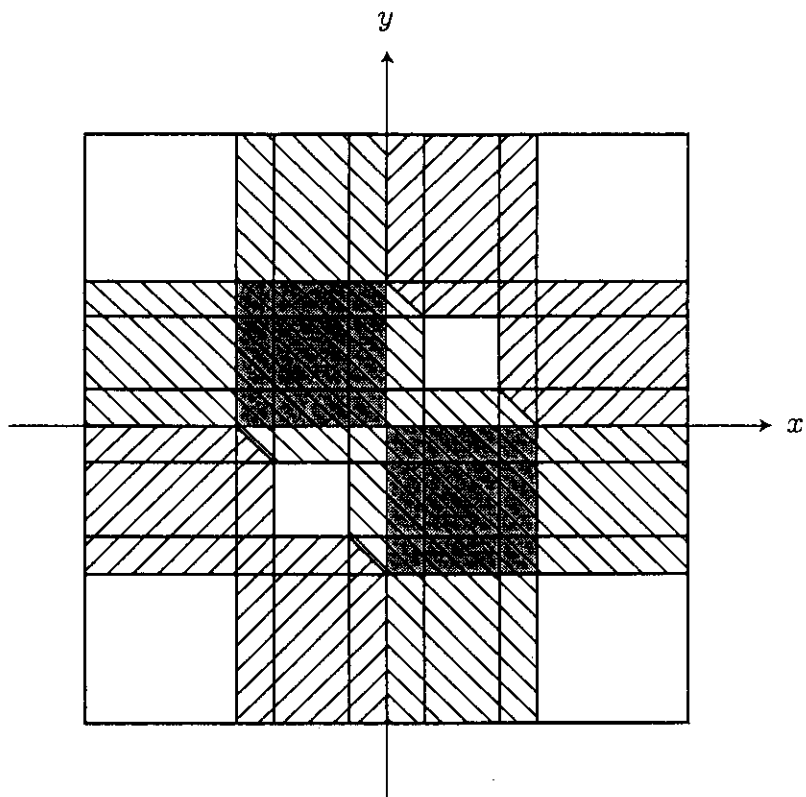


Figure 11: Continuous Alternative beats Discontinuous Alternative

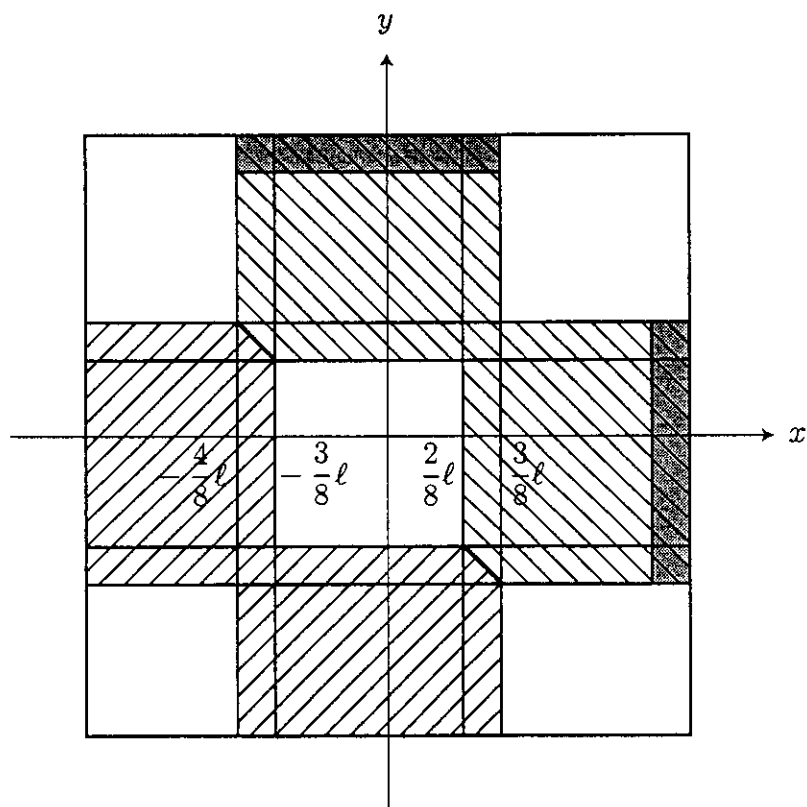


Figure 12: Symmetric Alternative beats Asymmetric Alternative

Table 1: Values of  $U(R_h : x, y)$ 

	$h \leq x$	$-h < x < h$	$x \leq -h$
$h \leq y$	$\beta h - \frac{\gamma}{l^2} h$	$\beta \left(-x + \frac{h}{2}\right) - \frac{\gamma}{l^2} h$	$-\frac{\gamma}{l^2} h$
$-h < y < h$	$\beta \left(y + \frac{h}{2}\right) - \frac{\gamma}{l^2} h$	$\beta  y - x  - \frac{\gamma}{l^2} h$	$\beta \left(-y + \frac{h}{2}\right) - \frac{\gamma}{l^2} h$
$y \leq -h$	$-\frac{\gamma}{l^2} h$	$\beta \left(x + \frac{h}{2}\right) - \frac{\gamma}{l^2} h$	$\beta h - \frac{\gamma}{l^2} h$

Table 2: Values of  $\frac{d}{dh} U(R_h : x, y)$ 

	$\frac{h}{2} \leq x$	$-\frac{h}{2} < x < \frac{h}{2}$	$x \leq -\frac{h}{2}$
$\frac{h}{2} \leq y$	$\beta - \frac{\gamma}{l^2}$	$\frac{\beta}{2} - \frac{\gamma}{l^2}$	$-\frac{\gamma}{l^2}$
$-\frac{h}{2} < y < \frac{h}{2}$	$\frac{\beta}{2} - \frac{\gamma}{l^2}$	$-\frac{\gamma}{l^2}$	$\frac{\beta}{2} - \frac{\gamma}{l^2}$
$y \leq -\frac{h}{2}$	$-\frac{\gamma}{l^2}$	$\frac{\beta}{2} - \frac{\gamma}{l^2}$	$\beta - \frac{\gamma}{l^2}$

Table 3: Sign of  $\frac{d}{dh} U(R_{\frac{l}{3}} : x, y)$  for  $0 \leq \gamma \leq \frac{1}{2}\beta l^2$ 

	$\frac{l}{3} \leq x$	$-\frac{l}{3} < x < \frac{l}{3}$	$x \leq -\frac{l}{3}$
$\frac{l}{3} \leq y$	+	+	-
$-\frac{l}{3} < y < \frac{l}{3}$	+	-	+
$y \leq -\frac{l}{3}$	-	+	+

Table 4: Sign of  $\frac{d}{dh} U(\phi : x, y)$  for  $\frac{1}{2}\beta l^2 \leq \gamma \leq \beta l^2$ 

	$0 \leq x$	$x \leq 0$
$0 \leq y$	+	-
$y \leq 0$	-	+